Journal of Statistical Physics, Vol. 11, No. 3, 1974

Measurements and Information for Thermodynamic Quantities

Göran Lindblad[1]

Received March 22, 1974

The relation between information and entropy for measurements of thermodynamic parameters is considered. The reduction in entropy that an observer can obtain in a system described by a fluctuating thermodynamic parameter is shown to be smaller than the information possessed by the observer. The information transfer and the entropy production due to the irreversible interaction of the observed system with the measuring instrument are compared.

KEY WORDS: Information; measurements; entropy; fluctuations; thermodynamics.

1. INTRODUCTION

It is well known that the process of measurement is of fundamental importance for the interpretation of quantum theory. This process introduces a noncausal and irreversible interaction ("reduction of the wave packet") between the quantum system S and the observer M, when the latter is considered as a macroscopic, classical system (the measuring instrument).[(1)] It is easily seen that this interaction is directly related to the information gain obtained by M about the value of the measured observables of S. In Ref. 2 it was shown that for a class of measurements slightly more general than the process considered by von Neumann[(1)] the information gain and the changes

[1] Department of Theoretical Physics, Royal Institute of Technology, Stockholm, Sweden.

in the statistical entropies of S and M satisfy certain inequalities. As the statistical entropy is invariant under unitary (reversible) transformations, an information gain will imply an irreversible development for S and M.

When one attempts to describe the interaction between S and M by treating M also as a quantum system, it is found necessary to consider an additional, external, observer M', or to introduce some extra assumption concerning M in order to achieve the "reduction of the wave packet" for the system $S + M$. This extra assumption can be of the type of the ergodic hypothesis or master equation that is commonly employed in statistical mechanics in order to pass from mechanics to thermodynamics. In fact, the reason for this assumption is essentially the same in both cases: The measurement process as well as the relaxation of macroscopic systems toward equilibrium both contain an irreversible element that cannot be described by the unitary time evolution for finite, closed, quantum systems.[3,4] It seems to be an interesting speculation that if the irreversible nature of the quantum measurement process resides in the macroscopic nature of the measuring instrument M, then some analogous phenomenon may exist already for measurements on the classical level of statistical mechanics. There does not seem to exist any general theory of measurements for macroscopic systems. The importance of the measurement process has been noted, however, in connection with the discussion of Maxwell's demon by Smoluchowski, Szilard, and Brillouin.[5–7] (The similarity between Maxwell's demon and the quantum measurement process was pointed out already by von Neumann.[1]) The demon was created in order to introduce an apparent contradiction to the second law of thermodynamics: By utilizing the statistical fluctuations in a system in thermodynamic equilibrium, the demon obtains work out of a heat reservoir via a cyclic process. Szilard[5] postulated that the gain of information by the demon, i.e., the establishment of a correlation between the state of the observed system and the memory of the demon, is accompanied by at least a proportional increase in thermodynamic entropy. He then shows for some simple systems the relation between the information content in the memory and the entropy decrease in the observed system that can be achieved (which is in its turn proportional to the available work).

Brillouin[6] attempted to derive the entropy increase during the process of gaining information from physical causes. In order to detect the fluctuations on a molecular level, Maxwell assumed that the demon was capable of "seeing" the molecules of a gas. Brillouin pointed out that in order to detect the molecules the demon could not use the isotropic blackbody radiation corresponding to the temperature of the gas. Instead he has to use a source of photons with a different spectrum, e.g., a black body of a higher temperature. The absorption of the photons by the demon, the gas, or the walls of the volume will then give rise to an entropy increase, which Brillouin claimed to be greater than the information gain multiplied by Boltzmann's constant. Unfortunately, this often repeated argument is incomplete, as is explained in Section 8. Since it is often accepted as an indication of the consistency of statistical mechanics with the second law of thermodynamics,[8] it seems worthwhile to try to save it. A detailed treatment is given in Section 8. There are several points in this type of argument that are open to doubts and disagreements[9]; in fact they are gedanken-experiments with a rather metaphysical flavor. It seems that the metaphysical details, like demons, are actually superfluous and that a discussion of measurements in the realm of statistical mechanics can be founded on more conventional arguments. This claim is based on the following fact. From Einstein's theory of fluctuations in thermodynamic systems it follows that, insofar as the theory is applicable to small systems, the average size of the fluctuations, measured by the decrease of the Boltzmann entropy from the equilibrium value, is of the order of k (Boltzmann's constant) independently of the size of the system. Hence there is no advantage in considering the artificial example of, e.g., a one-molecule gas. In fact the identification of the thermodynamic quantities is more dubious than for a large system. Hence, if there is any sense in this type of discussion at all, then it should be possible to make it general and applicable to measurements of fluctuations in an arbitrary thermodynamic system.

In order to connect the preceding remarks to the problem of measurement of thermodynamic quantities, we note that a fundamental feature of equilibrium and nonequilibrium statistical mechanics is that macroscopic deviations from equilibrium (e.g., during the relaxation to equilibrium after a change of boundary conditions) and spontaneous fluctuations in an equilibrium state are treated on the same footing. Thus the distribution of the microcanonical states in a canonical equilibrium state can be interpreted, for a sufficiently large system, as a distribution of macroscopic equilibrium states (see Section 4).[10,11] Similarly, in the derivation of the fluctuation-dissipation theorem an essential assumption is that the average regression of spontaneous fluctuations is identified with the macroscopic response of the system when an external force is removed.[12,13] Consequently, it is natural to attempt, as a first approximation, to treat the measurement of thermodynamic fluctuations as a special case of the measurement of macroscopic thermodynamic variables.

In Section 2 the fundamental notions of information theory are defined, especially those of entropy and relative entropy. Some properties of these quantities for Gibbs' canonical states and for normal probability distributions are given in Section 3. Note that the thermodynamic entropy is defined here (from the Gibbs' state) to be dimensionless through dividing by k. Then the connection between the quantity of information and the entropy reduction that can be achieved in S is discussed in a manner which is believed to be new and which employs thermodynamic quantities throughout (Section 5). The measurement of intensive thermodynamic parameters is treated in Sections 6 and 7, using linear nonequilibrium statistical mechanics, especially the fluctuation-dissipation theorem. Hence many measuring instruments that use nonlinear phenomena (e.g., for amplification) will be outside the discussion. It should be pointed out that exactly the latter type of instruments are of interest in the case of quantum measurements. It is found that under certain restrictive conditions the entropy production in the instrument is greater than the information gain, but that the inequality does not hold in the generality one may have wished.

2. INFORMATION AND ENTROPY

Let a measure space (Ω, Σ, μ) be given. If κ and ν are two probability measures on Ω, we define the relative (or conditional) entropy as

$$S(\kappa|\nu) = \int d\kappa \ln[d\kappa/d\nu]$$

when κ is absolutely continuous relative to ν (i.e., when the Radon–Nikodym derivative $d\kappa/d\nu$ exists) and as $+\infty$ otherwise. When κ and ν are absolutely continuous relative to μ we can write

$$S(\kappa|\nu) = S(f|g) = \int d\mu\, f(\omega) \ln[f(\omega)/g(\omega)]$$

where $f = d\kappa/d\mu$ and $g = d\nu/d\mu$. This quantity has also been called "the information gain when the distribution g is replaced by f"[14] and "the mean information for discrimination in favor of f against g."[15]

To a discrete random variable $X: \Omega \to R$ corresponds a partition $\{\xi_x\}$ of Ω: $\xi_x \equiv \{\omega; X(\omega) = x\}$. Define

$$S_X(\kappa|\nu) = \sum \kappa(\xi_x) \ln[\kappa(\xi_x)/\nu(\xi_x)]$$

If we denote the probability distributions of X in the states κ and ν by

$$p(x) = \kappa(\xi_x), \qquad q(x) = \nu(\xi_x)$$

we can write this as

$$S_X(\kappa|\nu) = S(p|q) = \sum p(x) \ln[p(x)/q(x)]$$

If X is a general ν-measurable real-valued random variable, put

$$F(x) = \kappa\{\omega; X(\omega) \leqslant x\}, \qquad G(x) = \nu\{\omega; X(\omega) \leqslant x\}$$

and define

$$S_X(\kappa|\nu) = \int dF(x) \ln[dF(x)/dG(x)]$$

If F and G are absolutely continuous relative to Lebesgue measure, then

$$S_X(\kappa|\nu) = S(f|g) = \int dx\, f(x) \ln[f(x)/g(x)]$$

where $f(x) = dF/dx$ and $g(x) = dG/dx$.

For finite quantum systems the analogous quantity is defined as follows: Given two states, i.e., two positive operators of trace 1, ρ and ρ', the relative entropy is defined by

$$S(\rho|\rho') = \mathrm{Tr}(\rho \ln \rho - \rho \ln \rho')$$

A more careful definition when the operator in the right-hand side is not of trace class is given in Ref. 2.

All the preceding definitions share the following properties: (i) $S(f|g) \geqslant 0$; equality holds if and only if $f = g$ a.e. (ii) $S(f|g)$ is invariant under coordinate transformations (unitary transformations in the quantum case).

The second property does not hold for the entropy of the distribution f:

$$S(f) = -\int d\mu(\omega)\, f(\omega) \ln f(\omega)$$

while the quantum mechanical entropy of a state

$$S(\rho) = -\mathrm{Tr}\, \rho \ln \rho$$

is still unitarily invariant.

Consider two random variables X, Y: $\Omega \to R$ on a probability space (Ω, Σ, μ). μ defines probability distributions (or densities in the continuous case) $p(x)$, $q(y)$, and $r(x, y)$ for X, Y, and (X, Y), respectively. Define the information between X and Y through (see, e.g., Ref. 16)

$$\begin{aligned} I(X, Y) &= \sum_{x,y} r(x, y) \ln[r(x, y)/p(x)q(y)] \\ &= \sum_{x} p(x) \sum q(y|x) \ln[q(y|x)/q(y)] \\ &= \sum p(x) S(q_x|q) \end{aligned} \tag{1}$$

where $q_x(y) = q(y|x) = r(x, y)/p(x)$ are the conditional probabilities which define a memoryless communication channel (the channel matrix).

Later X will represent a thermodynamic quantity which is to be measured and $q(y|x)$ will describe the statistical fluctuations in the response of the measuring instrument.

The channel capacity is defined as the maximum of $I(X, Y)$ with $q(y|x)$ fixed, $p(x)$ variable[16]

$$C = \sup_{p(x)} I(X, Y)$$

A simple rearrangement gives

$$\begin{aligned} I(X, Y) &= \sum p(x)S(q_x|q) = \sum p(x)S(q_x|q') - S(q|q') \\ &\leqslant \sum p(x)S(q_x|q') \end{aligned} \tag{2}$$

for any probability distribution q' such that $S(q|q')$ is finite. For example, we must have

$$I(X, Y) \leqslant \sum p_x S(q_x|q_{x'})$$

for all x'. If a certain value of C is desired, then this inequality gives certain conditions on the $q(y|x)$: For every x' we must have

$$S(q_x|q_{x'}) \geqslant C$$

for at least one x. This is a condition of "distinguishability" for the different distributions q_x.

3. PROPERTIES OF CANONICAL AND NORMAL DISTRIBUTIONS

Consider a finite quantum (or classical) system with Hamiltonian H. The Gibbs canonical state for the natural temperature $\beta = 1/kT$ is

$$\rho_\beta = \exp[-\beta H - \Phi(\beta)]$$

where

$$e^{\Phi(\beta)} = \operatorname{Tr} e^{-\beta H} = \int_0^\infty d\Omega(x)\, e^{-\beta x}$$

$\Omega(x)$ is the number of microstates with energy less than x. The thermodynamic energy and entropy are defined by

$$\begin{aligned} u(\beta) &\equiv \operatorname{Tr} \rho_\beta H = \int d\Omega(x)\, x e^{-\beta x - \Phi(\beta)} \\ s(\beta) &\equiv S(\rho_\beta) = -\operatorname{Tr} \rho_\beta \ln \rho_\beta = \beta u(\beta) + \Phi(\beta) \end{aligned}$$

Introduce

$$\begin{aligned} s(\beta|\beta') \equiv S(\rho_\beta|\rho_{\beta'}) &= \operatorname{Tr}(\rho_\beta \ln \rho_\beta - \rho_\beta \ln \rho_{\beta'}) \\ &= (\beta' - \beta)u(\beta) + \Phi(\beta') - \Phi(\beta) \\ &= \beta'[u(\beta) - u(\beta')] + s(\beta') - s(\beta) \\ &= \int_{\beta'}^{\beta} (\beta' - \beta'')\, d\mu(\beta'') \end{aligned}$$

$s(\beta|\beta')$ is the total increase in thermodynamic entropy when the system is brought into contact with a heat bath of natural temperature β' and allowed to attain equilibrium. For an arbitrary state ρ we have

$$S(\rho|\rho_\beta) = s(\beta) - S(\rho) + \beta[\rho(H) - u(\beta)]$$

where $\rho(H) \equiv \mathrm{Tr}\, \rho H$. If $\rho(H) = u(\beta)$, then

$$S(\rho) \leqslant s(\beta)$$

with equality iff $\rho = \rho_\beta$; hence the well-known fact that ρ_β is the state of given average energy $u(\beta)$ with the greatest entropy. Furthermore, the following relation holds:

$$S(\rho|\rho_\beta) = S(\rho|\rho_{\beta'}) + S(\rho_{\beta'}|\rho_\beta) + (\beta - \beta')[\rho(H) - u(\beta')]$$

From this it follows that if $\{\rho_i\}$ is given, $\lambda_i > 0, \sum \lambda_i = 1$, and $u(\beta') = \sum \lambda_i \rho_i(H)$, then

$$\sum_i \lambda_i [S(\rho_i|\rho_\beta) - S(\rho_i|\rho_{\beta'})] = s(\beta'|\beta) \tag{3}$$

Hence

$$\sum \lambda_i S(\rho_i|\rho_{\beta'}) \leqslant \sum \lambda_i S(\rho_i|\rho_\beta) \tag{4}$$

$$s(\beta'|\beta) \leqslant \sum \lambda_i S(\rho_i|\rho_\beta) \tag{5}$$

In (4) equality holds iff $\beta = \beta'$, in (5) iff $\rho_i = \rho_{\beta'}$ for all i.

The normal probability density on R^n with mean

$$a_i = \langle x_i \rangle$$

and symmetric, positive-definite covariance matrix

$$R_{ij} = \langle (x_i - a_i)(x_j - a_j) \rangle$$

is given by

$$g(x) = (2\pi)^{-n/2}(\det \mathbf{A})^{1/2} \exp\left[-\tfrac{1}{2} \sum A_{ij}(x_i - a_i)(x_j - a_j)\right]$$

where the matrix $\mathbf{A}$ is the inverse of $\mathbf{R}$. A simple calculation gives (in matrix notation)

$$S(g|g') = \tfrac{1}{2}[\ln(\det \mathbf{A}/\det \mathbf{A}') - n + \mathrm{Tr}\, \mathbf{A}'\cdot\mathbf{R} + (\mathbf{a} - \mathbf{a}')^{\mathrm{T}}\mathbf{A}'(\mathbf{a} - \mathbf{a}')] \tag{6}$$

Let f be an arbitrary and g a normal probability density in R^n. Define the normal densities g' and g'' through

$$\mathbf{R}(g') = \mathbf{R}(g), \qquad \mathbf{a}(g') = \mathbf{a}(f)$$
$$\mathbf{R}(g'') = \mathbf{R}(f), \qquad \mathbf{a}(g'') = \mathbf{a}(f)$$

Then a simple computation gives

$$S(f|g) = S(f|g') + S(g'|g) = S(f|g'') + S(g''|g)$$
$$= S(f|g'') + S(g''|g') + S(g'|g)$$

Hence

$$S(g''|g) \leqslant S(f|g), \qquad S(f|g'') \leqslant S(f|g') \leqslant S(f|g) \tag{7}$$

Furthermore,

$$S(f|g'') = S(g'') - S(f) \geqslant 0$$

i.e., the normal distribution has the greatest entropy of all distributions with given covariance matrix R.

4. FLUCTUATIONS IN THERMODYNAMIC PARAMETERS

For simplicity we will only consider temperature fluctuations here, but the generalization to arbitrary intensive thermodynamic parameters is trivial. The canonical state of natural temperature β is a superposition of microcanonical states of definite energy. Due to the great number of degrees of freedom of a macroscopic system, however, it is possible to associate a temperature to an isolated system with a definite energy. For a given energy u the temperature $\beta(u)$ is the unique solution of

$$u = -d\Phi(\beta)/d\beta$$

or, equivalently, as the value of β for which

$$\beta u + \Phi(\beta) \equiv s(\beta(u)|\beta) + s(\beta(u)) \tag{8}$$

has a minimum.[17,18] We can then interpret the canonical state as a probability density for macroscopic states indexed by β'[10]

$$f(\beta'|\beta) \equiv f_\beta(\beta')$$

defined by

$$f(\beta(x)|\beta)\, d\beta(x) = \exp[-\beta x - \Phi(\beta)]\, d\Omega(x) \tag{9}$$

The relative entropy of two such probability densities is the same as that of the two corresponding canonical states

$$S(f_\beta|f_{\beta'}) = s(\beta|\beta')$$

This follows from the invariance of the relative entropy under coordinate transformations. Of course the entropy of f_β is not equal to $s(\beta)$. From (8) it follows that

$$f_\beta(\beta(x))\, dx = \exp[-s(\beta(x)|\beta) - s(\beta(x))]\, d\Omega(x)$$

Introduce the Boltzmann (or microcanonical) entropy $s_B(x)$ through

$$s_B(x) = s(\beta) + \ln[\omega(x)/\omega(u(\beta))]$$

where $\omega(x) = d\Omega(x)/dx$. Note that we have chosen a definition which differs slightly from the conventional one by demanding that $s_B(u(\beta)) = s(\beta)$. For a sufficiently large system the canonical and microcanonical entropies should be approximately equal,

$$s_B(x) \simeq s(\beta(x))$$

If this approximation holds at least for $\beta(x) \simeq \beta$, then

$$f_\beta(\beta(x))\, dx \simeq C(\beta) \exp[-s(\beta(x)|\beta)]\, dx$$

where $C(\beta)$ is a normalization constant which does not depend on x. Hence the probability density for a fluctuation in temperature from β to β' is

$$f_\beta(\beta')\, du(\beta') \simeq C(\beta) \exp[-s(\beta'|\beta)]\, du(\beta')$$

which is essentially Einstein's formula. For $\beta' - \beta$ small

$$s(\beta'|\beta) \simeq -\tfrac{1}{2}(\beta' - \beta)^2\, \partial u/\partial\beta$$

which gives the Gaussian approximation for the fluctuations.

The expression (9) for f_β will be used in the following section. In Sections 6 and 7 the Gaussian form will be used for fluctuations (of extensive variables) in linear systems, where this approximation is natural.

A remark on the concept of temperature fluctuations is in order. Two notions of temperature are used here. The first is a parameter describing the canonical state, the second a random variable which is just a function of the energy. The advantage of the second concept over that of energy for large systems is that the extensive energy variable is transformed into an intensive one, which, when the size of the system goes to infinity, converges in some sense to the parameter value. There are of course other possible interpretations of temperature fluctuations as given by, e.g., Landau and Lifshitz[19] or Mandelbrot.[18]

5. ENTROPY REDUCTION FROM INFORMATION

We want to calculate the maximal reduction in the entropy that an external observer can achieve in a given thermodynamic system from a certain amount of information about the thermodynamic parameters. Equivalently, we can consider the maximal amount of work that can be extracted from the system in a given environment. This will be done by considering a simple but sufficiently general system: an infinite heat bath of temperature β_0, a finite but large system S (1) of temperature β_1, and a cylinder with an ideal gas enclosed by a piston (2). We assume that $\beta_1 \equiv X$ is a random variable with a probability density $f(x)$. The physical origin of this distribution can be left unspecified for the moment. Attached to (1) there is a measuring instrument (thermometer) M, the output of which is a random variable Y with a conditional probability density $g(y|x)$ describing the fluctuations.

As a result of one measurement of the temperature of (1) we obtain a value y of Y. From this value we want to give an estimate of the value of X which is optimal in some sense. If the temperature of (1) was perfectly known a priori, the maximal work could be obtained from it in the following way: The engine (2), originally in contact with the heat bath, is brought reversibly to the temperature β_1 by moving the piston, then the system (1) + (2) is brought back reversibly to the temperature β_0 in the same way. The work done by the system (1) + (2) is then

$$\Delta W = \beta_0^{-1} s_1(\beta_1|\beta_0)$$

When the information on X is incomplete, there will be losses due to the irreversible heat exchange which takes place between (1) and (2) if the estimated temperature β of (1) is not equal to the actual value. In fact there will be an entropy production

$$\Delta S' = s_1(\beta_1|\beta_1') + s_2(\beta|\beta_1')$$

where β_1' is the equilibrium temperature of (1) + (2). In the same way the system (1) + (2) cannot be brought back exactly to the temperature β_0 of the heat bath but ends up at a temperature β_0'. When (1) + (2) is brought into contact with the heat bath there is a further entropy increase

$$\Delta S'' = s_1(\beta_0'|\beta_0) + s_2(\beta_0'|\beta_0)$$

Obviously the total available work is given by

$$\beta_0 W = \Delta S = s_1(\beta_1|\beta_0) - \Delta S' - \Delta S''$$

The calculation is now much simplified if we assume that the heat capacity of (2) is much larger than that of (1). Then the temperature fluctuations in (2) can be neglected and we can make the approximations

$$\beta_1' \simeq \beta, \qquad \beta_0' \simeq \beta_0, \qquad \Delta S \simeq s_1(\beta_1|\beta_0) - s_1(\beta_1|\beta)$$

[the last approximate equality is actually exact when (1) has a constant heat capacity]. We find that $\beta \neq \beta_1$ implies a reduction in the available work as expected. Put

$$f(x|y) = g(y|x)f(x)/g(y), \qquad g(y) = \int g(y|x)f(x)\,dx$$

Then the average of ΔS given the value y of X is

$$\begin{aligned}\langle \Delta S\rangle_y &= \int \Delta S f(x|y)\,dx \\ &= \int [s_1(x|\beta_0) - s_1(x|\beta)]f(x|y)\,dx\end{aligned}$$

From (4) it follows that variation of β gives a maximum for $\langle \Delta S\rangle_y$ when $\beta = \beta_y$, where β_y is defined (uniquely) by

$$u_1(\beta_y) = \int u_1(x)f(x|y)\,dx \tag{10}$$

This is then the optimal estimate for the value of X when y is given. The maximum is, by (3),

$$\begin{aligned}\langle \Delta S\rangle_{y,\max} &= s_1(\beta_y|\beta_0) \\ &= s_1(\beta|\beta_0) + s_1(\beta_y|\beta) + (\beta_0 - \beta)[u_1(\beta_y) - u_1(\beta)]\end{aligned}$$

for all β. Let $\bar{\beta}$ be given by

$$u_1(\bar{\beta}) = \int u_1(\beta_y)g(y)\,dy = \int u_1(x)f(x)\,dx$$

Then

$$\begin{aligned}\langle \Delta S\rangle = \int g(y)s_1(\beta_y|\beta_0)\,dy &= s_1(\bar{\beta}|\beta_0) + \langle s_1(\beta_y|\bar{\beta})\rangle \\ &= s_1(\bar{\beta}|\beta_0) + s_1(\bar{\beta}) - \langle s_1(\beta_y)\rangle\end{aligned}$$

If we do not use the information contained in Y, then we can average ΔS over $f(x)$:

$$\int \Delta S f(x)\,dx = s_1(\bar{\beta}|\beta_0) - s_1(\bar{\beta}|\beta)$$

which has a maximum for $\beta = \bar{\beta}$ which is the predicted value of X from the a priori distribution $f(x)$, and the maximum is $s_1(\bar{\beta}|\beta_0)$. In the expression for ΔS above the term $s_1(\bar{\beta}|\beta_0)$ is then proportional to the available work with only the knowledge contained in the a priori distribution $f(x)$, and $\langle s_1(\beta_y|\bar{\beta})\rangle$ gives a contribution due to the additional information contained in Y.

Since (1) is a finite system, the energy (or, equivalently, the temperature as explained in the preceding section) exhibits fluctuations when the system is in thermal contact with a heat bath. If the system is isolated by the introduction of an adiabatic constraint, the value of the temperature is fixed and can be treated as a thermodynamic parameter. If $f(x)$ is now interpreted as the distribution of the temperature fluctuations in system (1), then the term $\langle s_1(\beta_y|\bar{\beta})\rangle$ is the average entropy decrease in the system obtained from the fluctuations using the information contained in Y. In order to compare this quantity with the amount of information

$$I(X, Y) = \int g(y) S(f(\cdot|y)|f(\cdot))$$

we note that $f(x)$ is derived from the canonical distribution with parameter $\bar{\beta}$. Due to Eq. (5) we have

$$\langle s_1(\beta_y|\bar{\beta})\rangle \leqslant I(X, Y)$$

6. THE MEASUREMENT OF THERMODYNAMIC QUANTITIES

We now consider the entropy production that is associated with the interaction of the measuring instrument M and the object system S. The set of n intensive thermodynamic parameters $\{X_i\}_1^n$ of S is to be measured. M is a thermodynamic system with $\{X_i\}$ acting as external forces driving the irreversible processes in M and with the extensive thermodynamic variables $\{Y_i\}_1^n$ as output. It seems obvious that the relation sought between information and entropy should be intimately connected to the existence of fluctuations in the quantities $\mathbf{Y}$. The fluctuations in $\mathbf{Y}$ determine the information about $\mathbf{X}$ that can be gained through one reading of the apparatus M, but they are also related to the response of M to external forces and to the entropy production through the fluctuation-dissipation theorem. It is an attractive conjecture that the irreversible processes due to the interaction always cause an entropy increase in the total system that is greater than the information gain.

In order to attempt to justify this conjecture, we will consider the case where the fluctuation-dissipation theorem is assumed to hold. For this it is necessary that M can be considered to be linear, i.e., the variations in the external forces are small enough to give a well-defined admittance function, and that S is much larger than M, so that the interaction of S and M does not influence the value of the parameters $\mathbf{X}$ appreciably. Furthermore, we assume that the parameters $\mathbf{X}$ are the generalized thermodynamic forces conjugate to the variables $\mathbf{Y}$ and that the random variables describing the fluctuations in $\mathbf{Y}$ are normal.(12,13)

Let the response of the system M to an external force $\mathbf{X}(t)$ be given by

$$\mathbf{Y}(t) = \langle\mathbf{Y}(t)\rangle + \boldsymbol{\eta}(t)$$

The relation between $\langle\mathbf{Y}(t)\rangle$ and $\mathbf{X}(t)$ is linear and causal, i.e., of the form (in matrix notation)

$$\langle\mathbf{Y}(t)\rangle = \int_0^\infty \mathbf{K}(\tau)\mathbf{X}(t - \tau)d\tau$$

Due to causality the admittance matrix

$$\hat{\mathbf{K}}(\omega) = \int_0^\infty \mathbf{K}(t)e^{i\omega t}\,dt$$

satisfies the Kramers–Kronig dispersion relations if $\hat{\mathbf{K}}(0)$ is finite. $\boldsymbol{\eta}(t)$ is a stationary normal random process (the thermal noise associated with **Y**) with an autocorrelation matrix

$$R_{ij}(\tau) \equiv \langle \eta_i(t)\eta_j(t + \tau)\rangle$$

and spectral density

$$\hat{\mathbf{R}}(\omega) \equiv \int_{-\infty}^\infty \mathbf{R}(t)e^{i\omega t}\,dt$$

Hence $\mathbf{Y}(t)$ has the probability density

$$f(\mathbf{y}) = (2\pi)^{-n/2}(\det \mathbf{R})^{-1/2} \exp\{-\tfrac{1}{2}[\mathbf{y} - \langle \mathbf{Y}(t)\rangle]^T\mathbf{R}^{-1}[\mathbf{y} - \langle \mathbf{Y}(t)\rangle]\}$$

where $\mathbf{R} \equiv \mathbf{R}(0)$ and T denotes the transposed matrix.

From the assumption that **X** and **Y** are conjugate variables and the equipartition property it follows that

$$\hat{\mathbf{K}}(0) = \beta\mathbf{R} \tag{11}$$

The fluctuation-dissipation theorem can be formulated in the following equivalent ways:

$$\beta\omega\hat{\mathbf{R}}(\omega) = \hat{\mathbf{K}}(-\omega) - \hat{\mathbf{K}}(\omega)^{\mathrm{T}}$$
$$\beta(d/dt)\mathbf{R}(t) = \mathbf{K}(-t) - \mathbf{K}(t)^{\mathrm{T}}$$

If $\mathbf{X}(\omega)$ is the Fourier transform of $\mathbf{X}(t)$, the total entropy production is given by

$$\Delta S = (4\pi)^{-1}\beta^2 \int \omega^2\mathbf{X}(\omega)^{\mathrm{T}}\hat{\mathbf{R}}(\omega)\tilde{\mathbf{X}}(\omega)\,d\omega \tag{12}$$

Let the state of S be characterized by the value **x** of the variable **X**, while the apparatus M is in equilibrium, with no external forces. At $t = 0$ the systems are brought into contact, which corresponds to

$$\mathbf{X}(t) = 0 \quad \text{for} \quad t < 0, \qquad \mathbf{X}(t) = \mathbf{x} \quad \text{for} \quad t \geqslant 0$$

Then the steady-state response of M is

$$\langle \mathbf{Y}(\infty)\rangle = \hat{\mathbf{K}}(0)\mathbf{x}$$

For the moment we will assume that $\hat{\mathbf{K}}(0)$ is finite. The entropy produced is

$$\Delta S = (4\pi)^{-1}\beta^2\mathbf{x}^{\mathrm{T}} \int \hat{\mathbf{R}}(\omega)\,d\omega\,\mathbf{x} = \tfrac{1}{2}\beta^2\mathbf{x}^{\mathrm{T}}\mathbf{R}\mathbf{x}$$

Suppose that some external observer measures the value of $\mathbf{Y}(t)$ at a sufficiently large time to obtain the steady-state response. The problem is then to distinguish the conditional probability densities

$$g_{\mathbf{x}}(\mathbf{y}) \equiv (2\pi)^{-n/2}(\det \mathbf{R})^{-1/2} \exp\{-\tfrac{1}{2}[\mathbf{y} - \hat{\mathbf{K}}(0)\mathbf{x}]^{\mathrm{T}}\mathbf{R}^{-1}[\mathbf{y} - \hat{\mathbf{K}}(0)\mathbf{x}]\}$$

corresponding to different values of $\mathbf{x}$. The relative entropy of two such distributions is given by (6):

$$S(g_{\mathbf{x}}|g_{\mathbf{x}'}) = \tfrac{1}{2}\beta^2(\mathbf{x} - \mathbf{x}')^{\mathrm{T}}\mathbf{R}(\mathbf{x} - \mathbf{x}')$$

where we have used (11) and the fact that $\mathbf{R}$ is a symmetric matrix. Hence we find that

$$\Delta S = S(g_{\mathbf{x}}|g_0)$$

Let $\mathbf{X}$ be a random variable with probability density $f(\mathbf{x})$ (note that the time dependence of $\mathbf{X}(t)$ is deterministic and given above). The information about $\mathbf{X}$ given by $\mathbf{Y}(\infty)$ is

$$I(\mathbf{X}, \mathbf{Y}(\infty)) = \int f(\mathbf{x})S(g_{\mathbf{x}}|h)\, d\mathbf{x}$$

where $h(\mathbf{y}) = \int f(\mathbf{x})g_{\mathbf{x}}(y)\, d\mathbf{x}$. For a given distribution $f(\mathbf{x})$ we are free to choose the initial (equilibrium) state of M by moving the zero of the $\mathbf{x}$ scale to some value $\bar{\mathbf{x}}$ in order to minimize the average entropy production:

$$\langle \Delta S \rangle = \int f(\mathbf{x})S(g_{\mathbf{x}}|g_{\bar{\mathbf{x}}})\, d\mathbf{x}$$

From (2) it follows that

$$I(\mathbf{X}, \mathbf{Y}(\infty)) = \langle \Delta S \rangle - S(h|g_{\mathbf{x}})$$

Hence

$$I(\mathbf{X}, \mathbf{Y}(\infty)) \leqslant \langle \Delta S \rangle$$

The minimum of $\langle \Delta S \rangle$ is achieved when $S(h|g_{\bar{\mathbf{x}}})$ has a minimum. According to (7) this is obtained when

$$\bar{\mathbf{y}} \equiv \int \mathbf{y}g_{\bar{\mathbf{x}}}(\mathbf{y})\, d\mathbf{y} = \int \mathbf{y}h(\mathbf{y})\, d\mathbf{y} = \hat{\mathbf{K}}(0) \int \mathbf{x}f(\mathbf{x})\, d\mathbf{x}$$

i.e., when

$$\bar{\mathbf{x}} = \int \mathbf{x}f(\mathbf{x})\, d\mathbf{x}$$

Now assume $\mathbf{X}$ to be normal with mean a and covariance matrix $\mathbf{Q}$. Then $\bar{\mathbf{x}} = \mathbf{a}$ and

$$\langle \Delta S \rangle = \langle S(g_{\mathbf{x}}|g_{\bar{\mathbf{x}}}) \rangle = \tfrac{1}{2}\beta^2 \operatorname{Tr} \mathbf{QR}$$

In this case one can actually compute the information. We find that $h(\mathbf{y})$ is normal with mean $\hat{\mathbf{K}}(0)\mathbf{a}$ and covariance matrix

$$\mathbf{R}' = \mathbf{R} + \hat{\mathbf{K}}(0)\mathbf{Q}\hat{\mathbf{K}}(0) = \mathbf{R}(\mathbf{1} + \beta^2\mathbf{QR})$$

Hence from (6)

$$\begin{aligned} S(g_{\mathbf{x}}|h) &= \tfrac{1}{2}\{\ln(\det \mathbf{R}'/\det \mathbf{R}) - \mathrm{Tr}[\hat{\mathbf{K}}(0)\mathbf{Q}\hat{\mathbf{K}}(0)\mathbf{R}'^{-1}] \\ &\quad + \mathbf{x}^{\mathrm{T}}\hat{\mathbf{K}}(0)\mathbf{R}'^{-1}\hat{\mathbf{K}}(0)\mathbf{x}\} \\ I(\mathbf{X}, \mathbf{Y}(\infty)) &= \langle S(g_{\mathbf{x}}|h)\rangle = \tfrac{1}{2}\ln(\det \mathbf{R}'/\det \mathbf{R}) \\ &= \tfrac{1}{2}\ln\det(1 + \beta^2\mathbf{QR}) \end{aligned} \tag{13}$$

At this point we can easily calculate the reduction in the entropy of S discussed in Section 5 if we assume that the distribution for $\mathbf{X}$ has its origin in the thermal fluctuations in S, i.e., if the normal distribution $f(\mathbf{x})$ is given by Einstein's fluctuation formula. This means that the entropy of S as a function of $\mathbf{x}$ is approximately

$$S_1(\mathbf{x}) = -\tfrac{1}{2}(\mathbf{x} - \mathbf{a})^{\mathrm{T}}\mathbf{Q}^{-1}(\mathbf{x} - \mathbf{a})$$

for some covariance matrix $\mathbf{Q}$. Bayes' formula gives the conditional probability density for $\mathbf{X}$ given $\mathbf{Y} = \mathbf{y}$ as the normal distribution with covariance matrix

$$\mathbf{Q}' = [\mathbf{Q}^{-1} + \hat{\mathbf{K}}(0)\mathbf{R}^{-1}\hat{\mathbf{K}}(0)]^{-1}$$

and mean

$$\mathbf{x}_{\mathbf{y}} = \mathbf{a} + \mathbf{Q}'\hat{\mathbf{K}}(0)\mathbf{R}^{-1}[\mathbf{y} - \hat{\mathbf{K}}(0)\mathbf{a}]$$

A calculation similar to that of Section 5, but using the normal distribution instead of the canonical one, gives that the predicted value of $\mathbf{X}$, given $\mathbf{Y} = \mathbf{y}$, is $\mathbf{x}_{\mathbf{y}}$ and that the average reduction achieved in the entropy of S is

$$\begin{aligned} S_1(\bar{\mathbf{x}}) - \langle S_1(\mathbf{x}_{\mathbf{y}})\rangle &= 0 + \tfrac{1}{2}\int h(\mathbf{y})(\mathbf{x}_{\mathbf{y}} - \mathbf{a})^{\mathrm{T}}\mathbf{Q}^{-1}(\mathbf{x}_{\mathbf{y}} - \mathbf{a})\, d\mathbf{y} \\ &= \tfrac{1}{2}\beta^2\,\mathrm{Tr}\,\mathbf{R}(\mathbf{1} + \beta^2\mathbf{RQ})\mathbf{Q}'\mathbf{Q}^{-1}\mathbf{Q}' \end{aligned} \tag{14}$$

Since the covariance of the intensive parameter $\mathbf{X}$ is inversely proportional to the size of S, and the covariance of the extensive quantity $\mathbf{Y}$ [and hence $\hat{\mathbf{K}}(0)$] is proportional to the size of M, the condition that S is much larger than M means that

$$\beta^2\mathbf{R} \ll \mathbf{Q}^{-1}$$

Inserted in (13) and (14), this gives that the information and the entropy reduction to first order of $\beta^2\mathbf{QR}$ are equal to $\tfrac{1}{2}\beta^2\,\mathrm{Tr}\,\mathbf{QR}$, which is equal to the entropy production.

For many interesting measuring systems the assumption behind the preceding discussion that the interacting systems reach an equilibrium state is not fulfilled, i.e., $\mathbf{R}$ is infinite. In this case we can only let S and M interact during a finite time τ. In order to carry out the derivations below it can be assumed, however, that $\mathbf{R}$ is large but finite. Let $\mathbf{X}$ have the value x, put

$$\mathbf{X}(t) = \mathbf{x} \quad \text{for} \quad t \in (0, \tau), \qquad \mathbf{X}(t) = \mathbf{0} \quad \text{elsewhere}$$
$$\mathbf{Z}(t) = \mathbf{Y}(t) - \mathbf{Y}(0) = \langle \mathbf{Y}(t) - \mathbf{Y}(0) \rangle + \boldsymbol{\eta}(t) - \boldsymbol{\eta}(0)$$

Then

$$\langle \mathbf{Z}(t) \rangle = \beta \mathbf{Q}(t)^{\mathrm{T}} \mathbf{x}$$

where

$$\mathbf{Q}(t) = \mathbf{R} - \mathbf{R}(t)$$

The covariance matrix of $\mathbf{Z}(t)$ is

$$\mathbf{R}_{\mathbf{Z}(t)} = \mathbf{Q}(t) + \mathbf{Q}(t)^{\mathrm{T}}$$

Let the value of $\mathbf{Y}$ be measured at $t = 0$ and $t = \tau$, and put $\mathbf{Q}(\tau) = \mathbf{Q}$ and $\mathbf{Z}(\tau) = \mathbf{Z}$. Let the normal distribution for $\mathbf{Z}$, given that the external force has the value $\mathbf{x}$, be denoted by $g_{\mathbf{x}}$.

The relative entropy of two such distributions is given by

$$S(g_{\mathbf{x}} | g_0) = \tfrac{1}{2}\beta^2 \mathbf{x}^{\mathrm{T}} \mathbf{Q}(\mathbf{Q} + \mathbf{Q}^{\mathrm{T}})^{-1} \mathbf{Q}^{\mathrm{T}} \mathbf{x}$$

The entropy produced in the total system is

$$\Delta S = \tfrac{1}{2}\beta^2 \mathbf{x}^{\mathrm{T}} (\mathbf{Q} + \mathbf{Q}^{\mathrm{T}}) \mathbf{x}$$

If we assume that the relevant variables are even under time reversal, that time reversal symmetry holds, and that there is no external magnetic field, then[13]

$$\mathbf{R}(\tau)^{\mathrm{T}} = \mathbf{R}(\tau)$$

Hence $\mathbf{Q}$ is symmetric and positive definite and

$$S(g_{\mathbf{x}} | g_0) = \tfrac{1}{4}\beta^2 \mathbf{x}^{\mathrm{T}} \mathbf{Q} \mathbf{x} = \tfrac{1}{4}\, \Delta S$$

For this special case, again using Eq. (2), one obtains the desired result: The entropy production is greater than the information gain.

As a simple example of a measuring instrument where the preceding inequalities fail to hold, we mention a fluxmeter, i.e., a galvanometer with torsion constant zero. In this case $\mathbf{R}$ is not finite, but a direct calculation of the matrix $\mathbf{Q}$ is simple. Let R' be the circuit resistance, K the mechanical damping, and G the flux linkage of the coil. If we neglect the inductance and the moment of inertia of the coil we get for $t \in (0, \tau)$

$$R'\dot{y}_1 + G\dot{y}_2 = x_1$$
$$-G\dot{y}_1 + K\dot{y}_2 = 0$$

where y_1 is the electric charge, y_2 the galvanometer deflection, and x_1 the potential difference to be measured. For τ large enough compared to the time constants defined by the inductance and moment of inertia we obtain

$$Q = \beta^{-1}\tau(KR' + G^2)^{-1}\begin{pmatrix} K & G \\ -G & R' \end{pmatrix}$$
$$\Delta S = \beta K(KR' + G^2)^{-1}x_1{}^2\tau$$
$$\langle(\Delta Z_2)^2\rangle = 2\beta^{-1}R'(KR' + G^2)^{-1}\tau$$

Assume that we measure only Z_2, i.e., the increase in the galvanometer deflection during the time τ. The relative entropy of the distributions for Z_2 corresponding to the external force x and zero is

$$S_2(g_x|g_0) = \beta G^2x^2\tau/4R'(KR' + G^2) = G^2\,\Delta S/4KR'$$

For $G^2 > 4KR'$ we can then obtain a counterexample to the general conjecture by a suitable choice of the probability distribution for $\mathbf{X}$.

Note that the variables $\mathbf{Y}$ perform a free Brownian motion without a natural zero point; hence the measurement of Y_2 at $t = 0$ is necessary. This also means that there is no energy associated with these coordinates. Hence a measurement of $\mathbf{Y}$ cannot, in turn, be considered to be a measurement of a macroscopic quantity.

7. INFORMATION RATE UNDER REPEATED MEASUREMENTS

The definition of the information about the value of $\mathbf{X}$ obtained via a measurement of $\mathbf{Y}$ as used in the preceding section applies to essentially a single "reading" of $\mathbf{Y}$. The amount of information can be increased through a repeated observation of $\mathbf{Y}$, which has the effect of averaging out the fluctuations.

Assume as before that $\hat{\mathbf{K}}(0)$ is finite and let

$$Y_x(t) = \hat{K}(0)x + \eta(t)$$

be the steady-state response of M if the force $X(t)$ has the constant value x. For simplicity we restrict ourselves to scalar X and Y. If Y^τ denotes the set of normal random variables $\{Y(t), t \in (0, \tau)\}$, define

$$S(Y_x{}^\tau|Y_0{}^\tau) = \lim_{n\to\infty} S(g_x^{(n)}|g_0^{(n)}) \tag{15}$$

where $g^{(n)}$ is the probability density of the set of random variables $Y(i\tau/n)$,

$i = 0, 1, \ldots, n - 1$. This is the relative entropy obtained by an infinite number of observations of Y during the time interval $(0, \tau)$. From (6) it follows that

$$S(g_x^{(n)} | g_0^{(n)}) = \left(\sum_{i,j=0}^{n-1} A_{ij}\right) \hat{K}(0)^2 x^2/2$$

where

$$(A^{-1})_{ij} = R[(i - j)\tau/n] \equiv \langle \eta(i\tau/n)\eta(j\tau/n)\rangle$$

The identity

$$\sum_{ijk} A_{ij} R[(j - k)\tau/n] = n$$

gives, if τ is much larger than the relaxation time of the process $\eta(t)$,

$$\sum_{ij} A_{ij} \sum_{k=-\infty}^{\infty} R(k\tau/n) \simeq n$$

Hence

$$\lim_{n\to\infty} \sum_{i,j=0}^{n-1} A_{ij} = \tau \Big/ \int_{-\infty}^{\infty} R(t)\, dt = \tau/\hat{R}(0)$$
$$\lim_{\tau\to\infty} \tau^{-1} S(Y_x^\tau | Y_0^\tau) = \hat{K}(0)^2 x^2/2\hat{R}(0)$$

This quantity can be calculated in an alternative way. Introduce a weight function φ through

$$\varphi(t) = \tau^{-1} \quad \text{for} \quad t \in (0, \tau), \qquad \varphi(t) = 0 \quad \text{elsewhere}$$

If τ is much larger than the relaxation time of the system, then

$$\langle Y_x(\varphi)\rangle \equiv \left\langle \int Y_x(t)\varphi(t)\, dt \right\rangle \approx \hat{K}(0)x$$
$$\langle \eta^2(\varphi)\rangle = (2\pi)^{-1} \int \hat{R}(\omega) \sin^2(\tau\omega/2)(\tau\omega/2)^{-2}\, d\omega$$
$$\simeq \tau^{-1}\hat{R}(0)$$

The relative entropy of the distributions for $Y_x(\varphi)$ and $Y_0(\varphi)$ is approximately

$$\hat{K}(0)^2 x^2 \tau/2\hat{R}(0)$$

Hence we obtain the same result as before. By averaging over a sufficiently long time we can thus increase the relative entropy, while the total increase in thermodynamic entropy remains constant (once equilibrium has been reached). Hence the conjectured inequality need no longer hold for every probability distribution for X. In order to carry out the averaging, however, we must use a second apparatus M' which performs an integration of the variable Y. Alternatively, one can use a number of external observer systems to record the value of Y at a sufficiently large number of points (one system for each point). Consequently, the entropy production in the integration system or in the memory systems must be considered, which of course means a return to the problem discussed in Section 6.

A related problem concerns the information rate and entropy production in a stationary Gaussian information channel. Let

$$Y(t) = \int K(t - \tau)X(\tau)\,d\tau + \eta(t)$$

where $X(t)$ and $\eta(t)$ are independent normal random processes. The information rate is defined by[20]

$$i(X, Y) = \lim_{\tau \to \infty} \tau^{-1} I(X^\tau, Y^\tau)$$

where $I(X^\tau, Y^\tau)$ is defined in analogy with (15). Let the spectral densities of $X(t)$ and $\eta(t)$ be $\hat{R}_X(\omega)$ and $\hat{R}_\eta(\omega)$ and let $\hat{K}(\omega)$ be related to $\hat{R}_\eta(\omega)$ by the fluctuation-dissipation theorem. Then[20]

$$i(X, Y) = (4\pi)^{-1} \int \ln[1 + \hat{R}_X(\omega)|\hat{K}(\omega)|^2/\hat{R}_\eta(\omega)]$$

On the other hand, the production of thermodynamic entropy per unit time in the system is given by

$$dS/dt = (4\pi)^{-1}\beta^2 \int \omega^2 \hat{R}_X(\omega)\hat{R}_\eta(\omega)\,d\omega$$

which follows from (12). In general we do not have

$$i(X, Y) \leqslant dS/dt$$

This inequality is, however, true for a system where $\hat{K}(\omega)$ is purely imaginary. It follows from the fluctuation-dissipation theorem that in this case

$$|\hat{K}(\omega)|^2 = \beta^2\omega^2\hat{R}_\eta(\omega)^2/4$$

and hence that

$$i(X, Y) \leqslant \tfrac{1}{4}\,dS/dt$$

There have been several attempts to make the information entropy relation plausible by using the Shannon–Hartley formula for the capacity of an information channel with Gaussian noise (see, e.g., Bell[21] or Pierce[22]). The crucial assumption is that the total signal energy is dissipated. From the discussion above it follows that these arguments are inconclusive, and that it is in fact possible to design systems such that the entropy production is smaller than the information transfer.

8. MAXWELL'S DEMON REEXORCISED

We will first give a short summary of the discussion of Szilard and Gabor's heat engine as given by Brillouin.[6]

A closed cylinder of volume V contains one gas molecule and is in contact with a heat reservoir of temperature T_0. The volume can be divided into two equal parts by a piston. An observer (Maxwell's demon) determines in which part the molecule happens to be and moves the piston reversibly toward the empty part. If the molecule is in thermal equilibrium with the walls of temperature T_0, this process yields a quantity of work

$$W = kT_0 \ln 2$$

A formal application of the formula for the entropy of an ideal gas gives an entropy decrease

$$\Delta S = -\ln 2$$

when the piston is inserted or when the position of the molecule is determined (the latter interpretation is more in line with the point of view taken here). This "gedanken-experiment" seems to defy the second law of thermodynamics. The essential question is, according to Brillouin: Is it actually possible for the demon to see the individual atoms? In the isotropic black-body radiation of the cylinder this is of course not so. But the observer can use a torch, i.e., a radiating black body of a temperature $T > T_0$, and a photocell to detect the photons. In Ref. 6 it is argued that the torch emits photons of median energy

$$h\nu = kT \ln 2$$

that the entropy increase in the gas and the photocell when one photon is absorbed is

$$\Delta S' = \beta_0 h\nu = (\ln 2)T/T_0$$

and that in order to distinguish the photons from the background radiation of temperature T_0 it is necessary to take $T > T_0$. The information I obtained when determining in which half-volume the molecule is located is $\ln 2$, i.e.,

$$\Delta S' \geqslant I$$

The entropy decrease achieved through the measurement is $\Delta S = -\ln 2$, hence $\Delta S + \Delta S' \geqslant 0$ and the second law is not violated after all.

The preceding argument is incomplete, as can be seen from the following remarks.

1. In the limit $T \to T_0$ it is evident that no information can be obtained, i.e., $I = 0$ in this case. Hence $T > T_0$ is not a sufficient condition.

2. The gas and the observer do not make up an isolated system. If the torch is included, we obtain

$$\Delta S' = (\beta_0 - \beta)h\nu = (\ln 2)(T - T_0)/T_0$$

3. There is a nonzero probability that a photon is emitted by the gas and absorbed by the torch. This leads to a decrease in entropy. In the limit $T = T_0$ there can be no entropy production, of course.

We easily realize that the essential point in the measurement process described above is the problem of how to distinguish between blackbody radiations of different temperatures. This of course leads us back to the measurement of temperature for macroscopic bodies. The problem can be formulated in the following way. An ideal black body of natural temperature β radiates in an enclosure of temperature β_0. Provided that an a priori distribution is given for β, what is the information about β contained in the measurement of the number of photons radiated in unit time, and how does it compare to the entropy increase due to the radiative transfer of energy?

The problem of how the detection of the photons actually takes place will be neglected, but it can be safely assumed that any real detection process gives an extra entropy increase, which is, however, in general difficult to calculate. In order to simplify the notation, we can consider one small frequency interval only (this can be realized by putting a filter between the black body and the rest of the system). When a black body is in equilibrium with its surroundings at temperate T the average rate of absorption or emission of photons in the interval $(\nu, \nu + d\nu)$ per unit area will be (Planck's law)

$$\langle N \rangle = 2\pi c^{-2}(e^{\beta h\nu} - 1)^{-1}\nu^2\, d\nu$$

The mean square fluctuation in the number of photons emitted or absorbed per unit time and area is

$$\langle (\Delta N)^2 \rangle = \langle N \rangle [1 + (e^{\beta h\nu} - 1)^{-1}]$$

This expresses the fact that the photons, due to their boson nature, do not behave as a stream of independent particles (in which case this quantity would be $\langle N \rangle$. But we can regard the radiation as consisting of a number $N_0 = 2\pi c^{-2}\nu^2\, d\nu$ of normal modes, each of which has mean $\langle n \rangle$ and variance $\langle n \rangle(1 + \langle n \rangle)$, where $\langle n \rangle = (e^{\beta h\nu} - 1)^{-1}$. The number of photons in a mode is geometrically distributed

$$p(n) = \langle n \rangle^n \langle n + 1 \rangle^{-n-1}$$

The relative entropy for two such distributions is

$$S(p_1|p_2) = \langle n\rangle_1 \ln[\langle n\rangle_1/\langle n\rangle_2] + (\langle n\rangle_1 + 1)\ln[\langle n+1\rangle_2/\langle n+1\rangle_1]$$

and we find

$$S(p_1|p_2) + S(p_2|p_1) = (\beta_2 - \beta_1)(\langle n\rangle_1 - \langle n\rangle_2)h\nu$$

Hence

$$S(p_1|p_2) \leqslant (\beta_2 - \beta_1)(\langle n\rangle_1 - \langle n\rangle_2)h\nu$$

When considering the nonequilibrium situation at hand, we must assume that the distribution of the emitted photons from a black body is independent of the temperature of the other bodies present, and that the distribution of the absorbed photons is independent of the temperature of the body itself. Then the right-hand side of the inequality above is the entropy increase per unit time due to one mode of the radiative energy transfer between two black bodies of temperature β_1 and β_2, respectively.

When the contributions of the N_0 modes are added as independent random variables the relative entropy is easily seen to be additive and so, of course, is the entropy production. If the random variables have a statistical dependence, the relative entropy is subadditive, while the entropy production is still additive, and hence the inequality is still valid.

Let the random variable X be the temperature of the body and Y the number of photons in the frequency range $(\nu, \nu + d\nu)$ radiated, during a unit time interval, from the body to the surroundings of temperature β_0. Then

$$I(X, Y) = \int f(\beta)S(p_\beta|p)\,d\beta \qquad \text{and} \qquad p(n) \equiv \int f(\beta)p_\beta(n)\,d\beta$$

From (2) it follows that

$$\begin{aligned} I(X, Y) &\leqslant \int f(\beta)S(p_\beta|p_{\beta_0})\,d\beta \\ &= \langle(\beta_0 - \beta)(\langle N\rangle_\beta - \langle N\rangle_{\beta_0})\rangle h\nu \end{aligned}$$

where the right-hand side is the entropy production due to the radiation transfer.

9. DISCUSSION

The relation between the entropy reduction achieved in the system S and the information content in the apparatus M discussed in Section 5 seems to be satisfactory as far as the interpretation of the temperature (or any intensive parameter) for an isolated system can be accepted. The choice of a temperature for an isolated system is not unique.[11,18,23] The formula for the optimal estimate (10), which reduces to the function $\beta(u)$ when the energy of S is known, shows that the definition used here is self-consistent. It also gives an interpretation of the temperature estimate that seems more physical than the "maximum likelihood" prescription[11]: The temperature estimate is chosen to maximize the available work. It has been claimed, in contradiction to the conclusion above, that there is no relation between the measure of information and the entropy, and that the observer need not have any information in order to obtain work from the system. It seems that the weak point of the argument presented in Ref. 9 lies in the lack of a concept of work as opposed to random fluctuations of the energy. In order to obtain work from the system it is by definition necessary to apply an external force of a macroscopic nature to the system. The resulting change in the conjugate coordinate is composed of a deterministic part (the mean value over the relevant ensemble) and of random fluctuations. The work is then defined as the average energy transfer over a large number of trials and hence the fluctuations are averaged out. In Section 5 it is understood that such an averaging has been made and hence that the fluctuations in all systems except the object system S can be neglected.

It may be of some interest to compare the results of Section 5 with those of Szilard.[24] Szilard calculates the work that can be obtained from an ensemble in a noncanonical state when it is assumed that this ensemble can be transformed into a canonical one by some special types of thermodynamic processes. The result can be expressed as the entropy of the given state relative to the canonical state of the same mean energy, which resembles the formulas of Section 5.

As has already been pointed out, the measurement of thermal fluctuations, once an adiabatic constraint has been introduced, is treated here as a special case of the measurement of macroscopic parameters. This method leads to some problems of interpretation. Let the total system $S + M$ obtain equilibrium and let M be much smaller than S. If the probability distribution for the parameters of S is due to the thermal fluctuations, then we saw in Section 6 that the entropy reduction achieved in S, the information gain, and the entropy production in M are approximately equal. This means that when the information gain is used to reduce the entropy of S the system $S + M$ seems to be approximately reversible. Now the following objection may be raised. Let S and M be treated on the same footing as finite systems and let them be in equilibrium with a heat bath. It then follows from the properties of the canonical state that the energies of S and M after being separated from the heat bath and brought into contact with each other are independent random variables.[24] Hence no information is contained in one of the variables about the other. Furthermore, no thermodynamic entropy is produced when the systems are brought into contact. This seems to contradict the assumption stated above. The resolution of the apparent paradox is as follows. If we let S and M be in thermal contact and if we repeatedly measure the energy of M via an external apparatus M' (which leaves the energy of M invariant, e.g., an ideal quantum measurement), then the information per measurement about the energy of the system S is approximately the same as that calculated by the formalism of Section 6. The entropy production, on the other hand, is calculated from the mean value of the energy of M, which will depend on the energy of S, and is generally different from zero.

The relation between information and entropy production in the instrument M treated in Sections 6 and 7 turned out to be less general than one could wish. In order to obtain the desired inequality, some further restrictions must be imposed on M or on the measurement procedure. One possibility is to prescribe that the system $S + M$ should be allowed to attain equilibrium. The fact that counterexamples to the general inequality exist does not seem to have been pointed out before. The interpretation of this fact is not clear and the first question is, of course, Could this be used to circumvent the second law of thermodynamics? In order to give a negative answer, we should prove that the information in the general case cannot be used by a macroscopic observer in order to manipulate the boundary conditions without further entropy increase. This is evidently true for the case of a fluxmeter discussed in Section 6, where the pointer performs a free Brownian motion. An external observer can use this information, e.g., through an observation via photons, which brings us back to the type of problems discussed in Section 8.

REFERENCES

1. J. von Neumann, Mathematical Foundations of Quantum Mechanics, Princeton University Press (1965).
2. G. Lindblad, *Commun. Math. Phys.* **33**:305–322 (1973).
3. A. Daneri, A. Loinger, and G. M. Prosperi, *Nucl. Phys.* **33**:297–319 (1962).
4. C. George, I. Prigogine, and L. Rosenfeld, *Dan. Vid. Selsk. Mat.-fys. Medd.* **38** (12) (1972).
5. L. Szilard, *Z. Phys.* **53**:840–856 (1929).
6. L. Brillouin, *Science and Information Theory*, Academic Press, New York (1962).
7. O. Penrose, *Foundations of Statistical Mechanics*, Pergamon Press, Oxford (1970).
8. A. B. Pippard, *The Elements of Classical Thermodynamics*, Cambridge University Press (1961).
9. J. M. Jauch and J. G. Baron, *Helv. Phys. Acta* **45**:220–232 (1972).
10. R. F. Greene and H. B. Callen, *Phys. Rev.* **83**:1231–1235 (1951).
11. L. Tisza and P. M. Quay, *Ann. Phys.* **25**:48–90 (1963).
12. H. B. Callen and R. F. Greene, *Phys. Rev.* **86**:702–710 (1952).
13. S. R. de Groot and P. Mazur, *Nonequilibrium Thermodynamics*, North-Holland, Amsterdam (1962).
14. A. Renyi, *Wahrscheinlichkeitsrechnung*, VEB Deutscher Verlag der Wissenschaften, Berlin (1966).

15. S. Kullback, *Information Theory and Statistics*, Wiley, New York (1959).
16. R. B. Ash, *Information Theory*, Interscience, New York (1965).
17. A. I. Khinchin, *Mathematical Foundations of Statistical Mechanics*, Dover, New York (1949).
18. B. Mandelbrot, *Ann. Math. Statist.* **34**:1021–1038 (1962).
19. L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Pergamon Press, Oxford (1959).
20. M. S. Pinsker, *Information and Information Stability of Random Variables and Processes*, Holden-Day, San Francisco (1964).
21. D. A. Bell, *Information Theory*, Pitman, London (1953).
22. J. R. Pierce, *Symbols, Signals, and Noise*, New York (1961).
23. B. Mandelbrot, *J. Math. Phys.* **5**:164–171 (1964).
24. L. Szilard, *Z. Phys.* **32**:753–788 (1925).